FULL WAVEFORM ANALYSIS OF A NON-UNIFORM AND DISPERSIVE TDR MEASUREMENT SYSTEM

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ABSTRACT

The spectral analysis and the concept of input impedance are used to develop a wave propagation model for a TDR measurement system that can account for dielectric dispersion of the material and non-uniform nature of the transmission line. The numerical procedure of the wave propagation model is relatively simple and becomes a powerful tool for parametric studies, data interpretations, and inverse analyses. The dielectric permittivity, the reference characteristic impedance, and the length of each uniform section characterize a non-uniform transmission line. Results indicate that inversion of these three parameters will result in definite non-uniqueness, but uniqueness can be achieved when one of the three parameters is known. The inversion of impedance profile from TDR waveforms with known dielectric properties can be used for data interpretation of rock and soil deformation; while the inversion of dielectric spectrum from TDR waveforms with known impedance can be used to study soil dielectric properties.

INTRODUCTION

Time domain reflectometry (TDR) has experienced dramatic developments over the past 20 years. It has become a valuable tool for the measurement of soil water content and monitoring of relative displacement in rocks and soils. The two major features of TDR waveforms used for data interpretation are the travel time of the TDR waveform in the probe for water content measurement and the voltage amplitude of the reflected waveform from relative displacement. However, much more information on the dielectric properties of the soil is contained in the measured TDR waveforms. In addition, the amplitude of the reflected waveform from shear displacement is affected by the cable loss and multiple reflections. To extract extra information when measuring dielectric properties of materials and to interpret the data more reasonably when monitoring relative displacement require a more fundamental understanding of the wave propagation in the TDR probe.

In a TDR system, there can be more than one type of transmission line. A 50-Ω cable connecting the cable tester and the TDR probes is a coaxial transmission line. The measurement probe could be either a coaxial line or multiple rod line. We also need a transitional device (or probe head) to connect the cable and measurement probe. Therefore, most likely, the TDR system consists of cable tester (e.g., Tektronix 1502B) and non-uniform transmission line as shown in Figure 1. Furthermore, the surrounding medium may vary along the measurement probe in the case where the soil column is not homogeneous along the line axis (z). And the cross-sectional dimension may vary along the coaxial cable when it is subject to relative deformation. If the cross-sectional dimensions of the line or the properties of the surrounding medium vary along the line axis, then the per-unit-length parameters in the transmission-line equation will be functions of the position variable, z. This makes the resulting partial differential equations very difficult to solve. Such transmission lines are said to be non-uniform lines.
To realistically model TDR waveforms, it is necessary to account for dielectric dispersion of material under test and non-uniform nature of a transmission line measurement system. Yanuka et al. [1988] presented a model that considers multiple reflections in a non-uniform transmission line but does not consider the frequency dependency of the material dielectric permittivity. Heimovaara [1994] used spectral analysis to account for dielectric dispersion but his method is only applicable to a uniform transmission line (i.e., a matched system and a homogeneous material). This paper presents an approach that formulates multiple reflections of multi-section TDR measurement system in frequency domain to take into account of dielectric dispersion of material under test and non-uniform nature of the transmission line. A numerical spectral algorithm is used to simulate wave propagation in a non-uniform and dispersive transmission line. This wave propagation model serves as a forward model in a model-based inversion to backcalculate impedance profile of a cable or to determine dielectric properties of insulating materials. Results of the simulation are compared to the actual TDR waveforms of some calibration materials.

DIELECTRIC DISPERSION MODEL

The dielectric permittivity is in general a complex number and a function of frequency:

\[ \varepsilon(f) = \varepsilon'(f) - j\varepsilon''(f) \]  

in which \( f \) is the frequency; \( \varepsilon' \) and \( \varepsilon'' \) are the real and imaginary part of permittivity, \( \varepsilon \). The real part of permittivity is often what we call the dielectric constant. It is a measure of how much energy from an external electric field is stored in a material. The imaginary part of permittivity is called the loss factor and is a measure of how dissipative or lossy a material is to an external electric field. It is convenient to define the equivalent permittivity \( \varepsilon^* = \varepsilon' - j(\varepsilon'' + \sigma / 2\pi f) \) to represent the total effect of permittivity, \( \varepsilon \), and conductivity, \( \sigma \). In terms of relative permittivity \( (\varepsilon_r = \varepsilon / \varepsilon_0) \), \( \varepsilon^* \) can be written as

\[ \varepsilon_r^* = \varepsilon' - j\left( \varepsilon'' + \frac{\sigma}{2\pi\varepsilon_0} \right) \]  

in which \( \varepsilon_0 \) is the dielectric permittivity of free space (equal to 8.854×10^{-12} F/m in MKS system). For a homogeneous material, the dielectric dispersion within TDR bandwidth may be described by Debye equation [Hasted, 1973].

\[ \varepsilon^*_r(f) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + jf/f_{rel}} - \frac{j\sigma}{2\pi\varepsilon_0} \]  

where \( \varepsilon_\infty \) is the real value of dielectric permittivity at \( f = 0 \), \( \varepsilon_s \) is the real value of dielectric permittivity at \( f \to \infty \), and \( f_{rel} \) is a modified average relaxation frequency related to the molecular polarization. For wet soils, the interaction between liquid phase and solid phase becomes significant when the particle size gets small. The resulting dielectric spectrum may be described more realistically by a sum of two or more different Debye's parameters having closely spaced relaxations (e.g., one for bound water and one for free water). In this study, some liquids whose dielectric permittivity can be described very well by Debye equation will be used to verify the wave propagation model.

SOLUTION OF TRANSMISSION LINE WAVE EQUATION

The interaction between electric and magnetic energy gives rise to the propagation of electromagnetic waves. Due to the special field structure (i.e., Transverse ElectroMagnetic Mode) inside the transmission line, line current and voltage between conductors can be uniquely defined and used to describe the electromagnetic wave propagation in a transmission line. The general solution to a uniform transmission line in phasor form is
where \( \hat{V} \) is the voltage between two conductors and \( \hat{I} \) is the line current. \( \gamma \) is the propagation constant (wave number), which is a function of dielectric permittivity. It can be written as

\[
\gamma = \frac{j2\pi f}{c}\sqrt{\varepsilon_r}\tag{5}
\]

in which \( c \) is the speed of light. \( Z_c \) is the characteristic impedance, which is a function of dielectric permittivity and cross-sectional geometry of the transmission line. It can be written as

\[
Z_c = Z_p / \sqrt{\varepsilon_r}
\]

where the reference characteristic impedance, \( Z_p \), is the characteristic impedance of the transmission line filled with air, which is only a function of the cross-sectional geometry of the transmission line. \( \hat{V}^+ \) and \( \hat{V}^- \) in (4) are two unknown constants in the general solution. Equation (4) can be interpreted as the sum of forward- and backward-traveling waves with 2 unknown coefficients.

In the case of a uniform transmission line, an equivalent circuit shown in Figure 2 can represent a TDR system. The line is terminated at the load end, \( z = l \), with a load impedance, \( Z_L \). At the source end, \( z = 0 \), an independent voltage source \( V_S \) and a source impedance, \( Z_S \), terminate the line. Thus, the boundary conditions are

\[
\hat{V}(0) = \hat{V}_S - Z_S \hat{I}(0) \tag{7a}
\]

\[
\hat{V}(l) = Z_L \hat{I}(l) \tag{7b}
\]

At this point, we could apply the boundary conditions to solve for unknown coefficients \( \hat{V}^+ \) and \( \hat{V}^- \) (two unknowns, two equations). Substituting the solution of \( \hat{V}^+ \) and \( \hat{V}^- \) back to (4) we can then solve for the voltage or current along the transmission line. The solution of special interest is the voltage at \( z = 0 \), which is the sampling voltage display on the TDR oscilloscope. However, in order to extend the usefulness of the solution, input impedance approach will be taken.

The concept of input impedance is similar to the equivalent stiffness in mechanical dynamics in which force and displacement are analogous to the voltage and current. As shown in Figure 2, the input impedance \( Z_{in}(z) \) is the equivalent impedance when looking into the circuit (i.e. the uniform transmission line) at position \( z \) [Magnusson et al., 1992]. It is defined as

\[
Z_{in}(z) = \frac{\hat{V}(z)}{\hat{I}(z)} = Z_c \frac{\hat{V}^+ e^{-\gamma z} + \hat{V}^- e^{\gamma z}}{\hat{V}^+ e^{-\gamma z} - \hat{V}^- e^{\gamma z}}
\]

If the input impedance at the source end (i.e. the impedance at the source end looking into the transmission line) can be determined, the sampling voltage \( \hat{V}(0) \) can be calculated by the boundary condition at the source end (7a).

First, the boundary condition at the load end (7b) can be written as
\[ Z_{in}(l) = \frac{\hat{V}(l)}{I(l)} = Z_L \]  

Then, the impedance at \( z = 0 \) can be derived as a function of the input impedance at \( z = l \) as (Lin et al. 2001)

\[ Z_{in}(0) = Z_c \frac{Z_{in}(l) + Z_e \tanh(y')} {Z_c + Z_{in}(l)\tanh(y')} \]  

where

\[ \tanh(y') = \frac{e^{y'} - e^{-y'}} {e^{y'} + e^{-y'}} = \frac{1 - e^{-2y'}} {1 + e^{-2y'}} \]  

Using the terminal condition at \( z = 0 \) and \( \hat{I}(0) = \hat{V}(0) / Z_{in}(0) \), \( \hat{V}(0) \) is determined as

\[ \hat{V}(0) = \frac{Z_{in}(0)}{Z_{in}(0) + Z_S} \hat{V}_S \]  

### TDR SYSTEM WITH A NON-UNIFORM TRANSMISSION LINE

In a general non-uniform transmission line, the per-unit-length line parameter will be functions of \( z \). In this case the transmission line differential equations become non-constant-coefficient differential equations. Although the differential equations remain linear (if the surrounding medium is linear), they are difficult to solve. However, if we can approximate the non-uniform line as a discretely uniform line as shown in Figure 3, (4) still represents the general solution for each uniform section.

For each of the \( n \) uniform sections, the general solution consists of the sum of forward and backward travelling waves with two unknown coefficients. Therefore, there are a total of \( 2n \) unknown coefficients (\( \hat{V}_i^+ \), \( \hat{V}_i^- \)). The terminal conditions are the same as in (7). This leaves \( 2n \) unknown coefficients with only two boundary conditions. The continuity constraints at the discontinuities between the terminations provide \( 2(n-1) \) more equations; and we could apply the \( 2n \) boundary conditions to solve for \( 2n \) unknown coefficients. However, the concept of input impedance enables a more systematic explicit procedure. Instead of solving the simultaneous equation, the simplified procedure involves calculating the input impedance from the end termination to the source termination. The analysis starts with the point farthest from the signal source, transforming the impedance back successively to the next discontinuity until the input is reached. This is done using (10) in a bottom-up fashion, which relates the impedance at the two ends of the uniform section of transmission line (Lin et al. 2001).
where $Z_{c,i}$, $\gamma_{i}$, and $l_{i}$ are the characteristic impedance, propagation constant, and length of each section. Once the input impedance looking into the entire line is obtained by use of (13), the sampling voltage $\hat{V}(0)$ can then be solved using (12).

Equations (13) and (12) provide the system function to simulate TDR waveforms of any TDR measurement system which may consist of different types of transmission lines and insulating materials. For a given TDR measurement system, we need to know the length $l_{i}$, the reference impedance $Z_{p}$, the equivalent dielectric permittivity $\varepsilon_{*}$ of each uniform section of the non-uniform transmission line, and the terminal impedances, $Z_{S}$ and $Z_{L}$ to predict the TDR waveform. Let the voltage source of the TDR be denoted by $V_{S}(t)$, the sampling voltage be denoted by $V_{TDR}(t)$, and the FFT algorithm by function $FFT()$. The simulation of a TDR waveform takes the following steps:

1. Determine appropriate window size for frequency and time to avoid aliasing in discrete Fourier Transform.
2. $\hat{V}_{S} = FFT(V_{S})$.
3. Determine $\hat{V}(0)$ from (13) and (12).
4. $V_{TDR}(t) = IFFT(\hat{V}(0))$.

**CALIBRATION OF TDR MEASUREMENT SYSTEM**

The calibration of transmission line parameters can be carried out by the model-based backward analysis. It can be seen from above that the properties of a non-uniform transmission line are the dielectric permittivity, the reference characteristic impedance, and the length of each uniform section. Two characteristics of a wave propagating in a non-uniform transmission line are the reflection/transmission at the interfaces of the mismatches and the propagation delay through each section. The reflection/transmission of waves depends on the impedance of the two sections adjacent to the interface, which are in turn functions of the reference characteristic impedance and dielectric permittivity of the insulating medium. The propagation delay is a function of the length of the section and the propagation constant, which is in turn a function of dielectric permittivity of the insulating medium. Different combinations of reference impedance and dielectric permittivity can result in the same impedance, while different combinations of section length and dielectric constant can result in the same propagation delay. Therefore, there are only two degrees of freedom in these three parameters. Considering the three characteristics of the non-uniform line together in the calibration will result in non-uniqueness.

The coaxial cable commonly used has nominal impedance of 50$\Omega$. However, The TDR pulses travelling on the coaxial cable are actually dispersive (i.e. the characteristic impedance is a function of frequency) and the rise time increases in duration substantially as the voltage pulse propagate along a cable. Long cables tend to filter out the higher frequencies of the TDR signal, thus reducing the bandwidth in the measurement [Heimovaara, 1993]. When monitoring soil/rock deformation, the reference characteristic impedance and frequency-dependent dielectric property of the cable can be calibrated by inversion from the measured waveform of a cable with known length. The frequency-dependent dielectric permittivity may be modeled by the Debye equation. As an example, the waveform of a 1m coaxial cable with open end is use to backcalculate the reference impedance and the dielectric property of the cable. The results shown that the cable has a reference impedance $Z_{p} = 71.24$ ohm and Debye's parameters $(\varepsilon_{s}, \varepsilon_{\infty}, f_{\sigma}, \frac{\sigma}{\varepsilon_{\infty}}) = (2.56, 2.04, 2.34$ MHz, 0 S/m). The comparison of the predicted and measured waveforms is shown in Figure 4. The slight difference between predicted waveform and the measured waveform are due to the BNC connector at the open end.
When measuring soil dielectric properties, a material with known dielectric properties is used for calibrating probe impedance and probe length. In this study, de-ionized water is used. The results of the system calibration were checked with measurements in materials with known dielectric properties listed in Table 1. The comparisons of the simulated and measured TDR waveforms for tap water, butanol alcohol, and ethanol alcohol are shown in Figure 5. It is observed that the simulated waveforms match the measured ones very well. This further validates the wave propagation model and results of the system calibration. The slight difference in the early part of the TDR waveforms (i.e. short-time response) reflects errors in high frequencies. This indicates that the higher order mode may have developed in addition to the TEM assumptions at high frequencies [Lin, 2001].

<table>
<thead>
<tr>
<th></th>
<th>Dielectric parameters</th>
<th>e</th>
<th>s</th>
<th>e_¥</th>
<th>f_{rel}, GHz</th>
<th>s_{dc}, S/m</th>
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<td>Expected</td>
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<td>3.30</td>
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<td></td>
<td>Estimated</td>
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<td>7.05</td>
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<tr>
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<td>Estimated</td>
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<td>3.54</td>
<td>15.500</td>
<td>0.0676</td>
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**DIELECTRIC SPECTRUM BY WAVEFORM MATCHING**

One can obtain the dielectric spectrum directly by solving the system function at each frequency. However, the accuracy and precision decreases as the frequency increases. It was observed that although the error in the system function at individual frequencies may be significant, the overall matches between the measured value and theoretical value in the system function and time-domain waveform are very good [Lin, 2001]. Therefore, it is possible to measure the dielectric spectrum by
waveform matching based on a dielectric model such as Debye equation.

The estimation of the dielectric parameters for the aqueous samples is listed in Table 1. The associated dielectric spectrum is compared to the expected spectra for butanol in Figure 6. For materials with relaxation frequency well within the TDR bandwidth, the inverse solution gives a very good estimate. When the relaxation frequency of the material such as water gets close to or beyond the TDR bandwidth, the uncertainty of estimation for $\varepsilon_\infty$ and $f_{rel}$ becomes significant. The inverse solution cannot resolve the dielectric spectra beyond the TDR bandwidth for a material with relaxation frequency higher than the TDR bandwidth. There are many different values of $\varepsilon_\infty$ and $f_{rel}$ that can give similar dielectric spectra within the TDR bandwidth. However, the inverse solution still provides a very good estimation of dielectric spectra within the TDR bandwidth.

![Graph showing expected vs. estimated dielectric spectra](image)

**Figure 6.** Measured dielectric spectrum of butanol alcohol from the estimation of Debye's parameters through waveform matching.

In general, the transmission line in a TDR measurement system is a non-uniform line and the dielectric permittivity of the insulating material may be frequency dependent. The spectral analysis and the concept of input impedance are used to develop a wave propagation model for a TDR measurement system that can account for dielectric dispersion of the material and non-uniform nature of the transmission line. The numerical procedure of the wave propagation model is relatively simple and becomes a powerful tool for parametric studies, data interpretation, and inverse analysis.

The dielectric permittivity, the reference characteristic impedance, and the length of each uniform section characterize a non-uniform transmission line. Results indicate that inversion of these three parameters will result in definite non-uniqueness, but uniqueness can be achieved when one of the three parameters is known. The reference impedance and the dielectric property of a cable with known length are calibrated with measured waveform of the coaxial cable with open end. The impedance and the length of a TDR probe is calibrated with measured TDR waveform in de-ionized water. The excellent match between the simulated and the measured waveforms verifies the wave propagation model. The inversion of impedance profile from TDR waveforms with known dielectric properties can be used for data interpretation of rock and soil deformation; while the inversion of dielectric spectrum from TDR waveforms with known reference characteristic impedance can be used to study soil dielectric properties.

**REFERENCE**


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